Technical Notes

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Static Solution of Monoclinic Laminated Circular Plates

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Introduction

OUTSIDE of rectangular plate and cylindrical shell configurations, little is known about the effects of lamination and material anisotropy. In this context, since the circular plate represents the limiting configuration of the crown section of numerous laminated shell of revolution types, to obtain a better understanding of the effects of lamination and material anisotropy in such zones, the present Note will study the mechanical fields of generally laminated circular plates. To date, except for a recent publication by Padovan and Lestingi,[1] analytical solutions for circular plates are available only for the special case of cylindrical orthotropy. With this in mind, the present Note will develop the solution of the governing equations of generally laminated monoclinic circular plates. To illustrate the effects of lamination as well as material anisotropy, Kirchhoff type theory is used.[2] Because of the generality of the solution procedure, all types of boundary conditions can be handled.

Plate Equations

For the present Note, the position of a point on the reference surface of a laminated circular plate is defined by r the radial distance from the origin and θ the circumferential coordinate. Considering the general case of unsymmetrically laminated monoclinic plates, the constitutive and plate stress equilibrium equations are given by[2]

$$\begin{bmatrix} N_{rr} \\ N_{\theta\theta} \\ N_{\theta r} \\ M_{rr} \\ M_{\theta\theta} \\ M_{\theta r} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} - \alpha_{rr} T_1 \\ \varepsilon_{\theta\theta} - \alpha_{\theta\theta} T_1 \\ \varepsilon_{\theta r} - \alpha_{\theta r} T_1 \\ \kappa_{rr} - \beta_{rr} T_2 \\ \kappa_{\theta\theta} - \beta_{\theta\theta} T_2 \\ \kappa_{\theta r} - \beta_{\theta r} T_2 \end{bmatrix} \tag{1}$$

$$N_{rr,r} + 1/rN_{r\theta,\theta} + (N_{rr} - N_{\theta\theta})/r + f_1 = 0 \tag{2}$$

$$N_{r\theta,r} + 1/rN_{\theta\theta,\theta} + 2/rN_{r\theta} + f_2 = 0 \tag{3}$$

$$M_{rr,rr} + 2/rM_{rr,r} + 1/r^2 M_{\theta\theta,\theta\theta} - 1/rM_{\theta\theta,r} + 2/rM_{\theta r,\theta r} + 2/r^2 M_{r\theta,\theta} + f_3 = 0 \tag{4}$$

where

$$\varepsilon_{rr} = u_{,r} \tag{5a}$$

$$\varepsilon_{\theta\theta} = 1/r(v_{,\theta} + u) \tag{5b}$$

$$\varepsilon_{r\theta} = 1/r(u_{,\theta} - v) + v_{,r} \tag{5c}$$

$$\kappa_{rr} = -w_{,rr} \tag{5d}$$

$$\kappa_{\theta\theta} = -(1/rw_{,r} + 1/r^2 w_{,\theta\theta}) \tag{5e}$$

$$\kappa_{r\theta} = 2(-1/rw_{,r\theta} + 1/r^2 w_{,\theta}) \tag{5f}$$

with $N_{rr}, \ldots, M_{rr}, \ldots$ being the usual plate resultants; A_{lk}, B_{lk}, D_{lk}, the material stiffnesses; α_{lk}, β_{lk}, related to the thermal coefficients of expansion; $\varepsilon_{rr}, \ldots, \kappa_{rr}, \ldots$, the classical plate strains; u, v, and w, the radial, circumferential, and lateral plate displacements; f_i, the plate loadings; T_i, the integrated plate temperatures; and ( ), partial differentiation. The boundary conditions associated with Eqs. (1–5) at $r = r_i$ (inside radius) and $r = r_o$ (outside) are

$$N_{rr} + k_1 u = \tilde{N}_{rr} \tag{6a}$$

$$N + k_2 v = \tilde{N} \tag{6b}$$

$$M_{rr} + k_3 w_{,r} = \tilde{M}_{rr} \tag{6c}$$

$$Q + k_4 w = \tilde{Q} \tag{6d}$$

where the overwavy bar denotes prescribed quantities, k_i can be so chosen as to yield traction or displacement boundary conditions, and N and Q are effective quantities.

Solution

Recasting Eqs. (1–5) in displacement form yields

$$A_{11}u_{,rr} + \cdots + A_{13}v_{,rr} + \cdots - B_{11}w_{,rrr} + \cdots + f_1 - (\alpha_{rr} T_{1,r} + \alpha_{r\theta}/rT_{1,\theta} + (\alpha_{rr} - \alpha_{\theta\theta})/rT_1) = 0 \tag{7}$$

$$A_{13}u_{,rr} + \cdots + A_{33}v_{,rr} + \cdots - B_{13}w_{,rrr} + \cdots + f_2 - (\alpha_{r\theta} T_{1,r} + \alpha_{\theta\theta}/rT_{1,\theta} + 2\alpha_{r\theta}/rT_1) = 0 \tag{8}$$

$$B_{11}u_{,rrr} + \cdots + B_{13}v_{,rrr} + \cdots - D_{11}w_{,rrrr} + \cdots + f_3 - (\beta_{rr} T_{2,rr} + 2\beta_{rr}/rT_{2,r} + \beta_{\theta\theta}/r^2 T_{2,\theta\theta} - \beta_{\theta\theta}/rT_{2,r} + 2\beta_{r\theta}/rT_{2,\theta r} + 2\beta_{r\theta}/r^2 T_{2,\theta}) = 0 \tag{9}$$

Because of the occurrence of the higher order derivatives of w in the r variable appearing in Eqs. (7–9), the standard Euler transformation in variables cannot be accomplished. This difficulty can be circumvented by introducing the functional transformation

$$w = r\eta \tag{10}$$

for $r > 0$. In terms of Eq. (10), Eqs. (7–9) can be rewritten in the following matrix form:

$$r^4 C_1 \zeta_{,rrrr} + r^3 C_2 \zeta_{,rrr\theta} + r^2 C_3 \zeta_{,rr\theta\theta} + rC_4 \zeta_{,r\theta\theta\theta} + C_5 \zeta_{,\theta\theta\theta\theta} + r^3 C_6 \zeta_{,rrr} + r^2 C_7 \zeta_{,rr\theta} + rC_8 \zeta_{,r\theta\theta} + C_9 \zeta_{,\theta\theta\theta} + r^2 C_{10} \zeta_{,rr} + rC_{11} \zeta_{,r\theta} + C_{12} \zeta_{,\theta\theta} + rC_{13} \zeta_{,r} + C_{14} \zeta_{,\theta} + C_{15} \zeta + \mathbf{f} - \gamma(T) = \mathbf{0} \tag{11}$$

where C_i are three by three matrices related to the A_{lk}, B_{lk}, and D_{lk} stiffnesses and

$$\zeta = \langle u, v, \eta \rangle^T \tag{12}$$

$$\mathbf{f} = \langle r^2 f_1, r^2 f_2, r^3 f_3 \rangle^T \tag{13}$$

$$\gamma(T) = \begin{bmatrix} r^2[\alpha_{rr} T_{1,r} + \alpha_{r\theta}/rT_{1,\theta} + (\alpha_{rr} - \alpha_{\theta\theta})/rT_1] \\ r^2(\alpha_{r\theta} T_{1,r} + \alpha_{\theta\theta}/rT_{1,\theta} + 2\alpha_{r\theta}/rT_1) \\ r^3[\beta_{rr} T_{2,rr} + (2\beta_{rr} - \beta_{\theta\theta})/rT_{2,r} + 2\beta_{r\theta}/rT_{2,\theta r} + \beta_{\theta\theta}/r^2 T_{2,\theta\theta} + 2\beta_{r\theta}/r^2 T_{2,\theta}] \end{bmatrix} \tag{14}$$

Now, introducing the change in variables

$$r = e^x \tag{15}$$

Eq. (11) can be reduced to

$$G_1 \zeta_{,xxxx} + G_2 \zeta_{,xxx\theta} + G_3 \zeta_{,xx\theta\theta} + G_4 \zeta_{,x\theta\theta\theta} + G_5 \zeta_{,\theta\theta\theta\theta} + G_6 \zeta_{,xxx} + G_7 \zeta_{,xx\theta} + G_8 \zeta_{,x\theta\theta} + G_9 \zeta_{,\theta\theta\theta} + G_{10} \zeta_{,xx} + G_{11} \zeta_{,x\theta} + G_{12} \zeta_{,\theta\theta} + G_{13} \zeta_{,x} + G_{14} \zeta_{,\theta} + G_{15} \zeta + \mathbf{f} + \gamma^*(T_i) = \mathbf{0} \tag{16}$$

Since Eq. (16) is noncanonical, assuming that T_i, $\mathbf{f}$, and $\tilde{N}_{rr} \ldots$ satisfy Dirichlet's conditions for the θ variable, following

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Padovan,[3] we can transform Eqs. (6) and (16) through the use of the usual finite Fourier exponential transform. Hence

$$\int_0^{2\pi} \{\text{Eqs.}(6, 16)\}\, e^{-jm\theta}\, d\theta \rightarrow \tag{17}$$

at $r = r_o, r_i$

$$N_{rrm} + k_1 u_m = \tilde{N}_{rrm} \tag{18a}$$
$$N_m + k_2 v_m = \tilde{N}_m \tag{18b}$$
$$M_{rrm} + k_3 w_{m,r} = \tilde{M}_{rrm} \tag{18c}$$
$$Q_m + k_4 w_m = \tilde{Q}_m \tag{18d}$$

and

$$G_1\zeta_{m,xxxx} + (G_6 + jmG_2)\zeta_{m,xxx} + (G_{10} - m^2G_3 + jmG_7)\zeta_{m,xx} + [G_{13} - m^2G_8 + j(-m^3G_4 + mG_{11})]\zeta_{m,x} + [G_{15} + m^4G_5 - m^2G_{12} + j(-m^3G_9 + mG_{14})]\zeta + \mathbf{f}_m + \gamma_m{}^*(T_{im}) = \mathbf{0} \tag{19}$$

where $j = (-1)^{1/2}$ and

$$\langle \zeta_m, N_{rrm}, \ldots, \tilde{M}_{rrm}, \mathbf{f}_m, T_{im} \rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \zeta, N_{rr}, \ldots, \tilde{M}_{rr}, \mathbf{f}, T_i \rangle\, e^{-jm\theta}\, d\theta \tag{20}$$

such that

$$\langle \zeta, \ldots \rangle = \sum_{m=-\infty}^{\infty} \langle \zeta_m, \ldots \rangle\, e^{jm\theta} \tag{21}$$

In spite of its complex form, the homogeneous solution of Eq. (19) can be taken as

$$\zeta_m(x) = \sum_{i=1}^{8} \alpha_{im}\, \xi_{im}\, e^{\lambda_{im} x} \tag{22}$$

where the latent roots λ_{im} and their associated latent vectors ξ_{im} must satisfy the following complex polynomial matrix:

$$\{\lambda_{im}{}^4 G_1 + \lambda_{im}{}^3(G_6 + jmG_2) + \lambda_{im}{}^2(G_{10} - m^2G_3 + jmG_7) + \lambda_{im}[G_{13} - m^2G_8 + j(-m^3G_4 + mG_{11})] + [G_{15} + m^4G_5 - m^2G_{12} + j(-m^3G_9 + mG_{14})]\}\xi_{im} = \mathbf{0} \tag{23}$$

In particular, λ_{im} must satisfy the complex characteristic polynominal of the pencil of Eq. (23), namely

$$\sum_{i=0}^{8} q_{im}\, \lambda_{im}{}^{(8-i)} = 0 \tag{24}$$

where q_{im} are complex constants.

The constants α_{im} appearing in Eq. (22) are obtained by satisfying the transformed boundary conditions denoted by Eq. (18). Thus, in terms of Eq. (22), the homogeneous solution of Eqs. (1–6) is given by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{m=-\infty}^{\infty} \sum_{i=1}^{8} \alpha_{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} \xi_{im}\, r^{\lambda_{im}}\, e^{jm\theta} \tag{25}$$

Note for the balanced case, Eq. (25) reduces to the solution of Ref. 1.

Numerical Results

To illustrate the substantial effects of laminate configuration as well as local material orientation β (see Fig. 1), the following boundary value problem was chosen

$$r = 7.5 \text{ in., } 36 \text{ in.}; \; u, v, w, M_\theta \equiv 0$$
$$f_3 = \sigma \cos\theta \quad \text{(lateral pressure)} \tag{26}$$

Since composite shells of revolution are typically spirally wound composites, the plate configuration studied herein will be considered fiber reinforced with logarithmic fiber loci. Figure 1 illustrates the significant effects of fiber orientation on the mechanical fields of a symmetrically and alternately plied four layer plate. The $\beta = 0°$ configuration illustrated can be used as a reference base from which to measure the significant effects of lamination and material anisotropy. It should be noted that for this case $A_{16}, \ldots, D_{26}$ are zero, hence the various mechanical

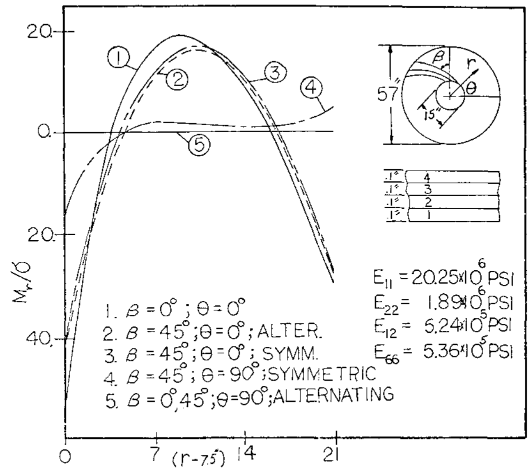

Fig. 1 Effects of β on the M_r field of an alternately and symmetrically plied four-layer simply supported circular plate.

fields are either even or odd. For example, M_r is even, hence $M_r|_{\theta=\pi/2} \equiv 0$.

For the alternately plied configuration, independent of $\beta\varepsilon(0, \pi/2)$, the bending inplane fields are coupled. Furthermore since $A_{16}, \ldots, D_{26}$ are zero, the anisotropic effects of fiber orientation are cancelled. That such is not the case for the symmetrically plied case is clearly seen from the nonzero $M_r|_{\theta=\pi/2}$ field depicted in Fig. 1.

Hence, unlike the alternately plied case, fiber orientation can cause significant asymmetries in the mechanical fields of symmetrically plied laminates. In summary, it should be noted that the analysis given herein can be used to study the effects of laminate configuration and material anisotropy for the generally laminated case.

References

[1] Padovan, J. and Lestingi, J., "Static Solution of Monoclinic Circular Plates," *AIAA Journal*, Vol. 9, No. 12, Dec. 1971, pp. 2473–2475.

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On the Dynamic Buckling of Shells of Revolution

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I. Introduction

IN Ref. 1, a series of experiments on the dynamic buckling of thin shallow spherical shells was carried out. The loads on the shell were generated by the rarefaction wave of a shock tube and had very sharp rise times (low μsec). The magnitudes of loading were less than 0.5 psi. The duration of loads on the shell

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